Math 270: Differential Equations Day 9 Part 1

Definitions and Results Before We Begin Sections 4.2 and 4.3

Definitions and Results Before We Begin Sections 4.2 and 4.3

Recall

Definition: The highest derivative appearing on the unknown function in the differential equation is called the order of the differential equation

Ex: What is the order of the differential equation $y''' - 3xy' + xy^2 = y''$?

Definition: The differential equation is linear if it can be written in the following form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = F(x)$$

Ex of a Linear DE:

Definitions and Results Before We Begin Sections 4.2 and 4.3

“Goal”: To Solve All Linear DEs

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = F(x)$$

We first focus on differential equations where the right side is 0.

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Definition: A linear differential equation is called homogeneous if it looks like…

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = \mathbf{0}$$

Result: The set of all solutions to a homogeneous *n*-th order linear differential equation is a vector space of dimension *n*

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Result: The set of all solutions to a homogeneous *n*-th order linear differential equation is a vector space of dimension *n*

1) If y_1 is a solution to a homogeneous linear differential equation and c is a scalar, then cy_1 is also a solution

2) If y_1 and y_2 are solutions to a homogeneous linear differential equation, then $y_1 + y_2$ is also a solution

3) If $y_1, y_2, \ldots y_k$ are solutions to a homogeneous linear differential equation and $c_1, c_2, \ldots c_k$ are scalars, then $c_1 y_1 + c_2 y_2 + \cdots + c_k y_k$ is also a solution

What does this mean?

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 1: $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = sin\, x$, and $y_4 = cos\, x$ are all solutions to the linear homogeneous differential equation $y^{(4)} - y = 0$. What are some other solutions to this differential equation?

1) If y_1 is a solution to a homogeneous linear differential equation and c is a scalar, then cy_1 is also a solution

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 1: $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = sin\, x$, and $y_4 = cos\, x$ are all solutions to the linear homogeneous differential equation $y^{(4)} - y = 0$. What are some other solutions to this differential equation?

2) If y_1 and y_2 are solutions to a homogeneous linear differential equation, then $y_1 + y_2$ is also a solution

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 1: $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = sin\, x$, and $y_4 = cos\, x$ are all solutions to the linear homogeneous differential equation $y^{(4)} - y = 0$. What are some other solutions to this differential equation?

3) If $y_1, y_2, \ldots y_k$ are solutions to a homogeneous linear differential equation and $c_1, c_2, \ldots c_k$ are scalars, then $c_1y_1 + c_2y_2 + \cdots + c_ky_k$ is also a solution

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 2: Prove that if y_1 and y_2 are solutions to the linear homogeneous differential equation $y'' - x^2y' + 2y = 0$ and c_1 and c_2 are scalars, then $c_1y_1 + c_2y_2$ is also a solution.

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Result: The set of all solutions to a homogeneous *n*-th order linear differential equation is a vector space of dimension *n*

3) If $y_1, y_2, \ldots y_k$ are solutions to a homogeneous linear differential equation and $c_1, c_2, \ldots c_k$ are scalars, then $c_1 y_1 + c_2 y_2 + \cdots + c_k y_k$ is also a solution

So…

Any linear combination of solutions to a homogeneous linear differential equation is also a solution to the equation

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Result: The set of all solutions to a homogeneous *n*-th order linear differential equation is a vector space of dimension *n*

Before we get to the "dimension *n*" part, we need to understand what independent solutions are   discuss

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

What does it mean for solutions to a homogeneous DE to be independent?

discuss 2, then 3, then many, then def

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Definition:

- Solutions $y_1, y_2, \ldots, y_k$ to a homogeneous linear DE are independent if the only linear of them that gives you the zero function is if all the scalars are 0.
- That is, if scalars $c_1, c_2, \ldots . c_k$ exist such that $c_1 y_1 + c_2 y_2 + \cdots + c_k y_k = 0$, then $c_1 = c_2 = \cdots = c_k = 0$
- For 2 solutions (k=2), they are independent if neither is a constant multiple of the other

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

How do you check if solutions $y_1, y_2, \ldots, y_k$ are independent?

- For 2 solutions (k=2), they are independent if neither is a constant multiple of the other
- Set up the equation $c_1 y_1 + c_2 y_2 + \cdots + c_k y_k = 0$ and if you can prove that all the coefficients are equal to zero $(c_1 = c_2 = \cdots = c_k = 0)$ then the solutions are independent
- Calculate the Wronskian and if it's never equal to zero, then the solutions are independent. If the Wronksian equals 0 at even one point, then the solutions are dependent.

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_k \\ y_1' & y_2' & \cdots & y_k' \\ y_1'' & y_2'' & \cdots & y_k'' \\ & \vdots & & \\ y_1^{(k-1)} & y_2^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix}$$

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 3: Are $y_1 = e^x$ and $y_2 = e^{4x}$ independent? Show in 3 ways.

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Note: $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$, ... , $y_k = e^{r_k x}$ are independent if the numbers $r_1, r_2, \ldots, r_k$ are all different.

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 4: Are $y_1 = \sin^2 x$, $y_2 = 1$, and $y_3 = \cos^2 x$ independent?

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 5: Are $y_1 = e^{2x}$, $y_2 = xe^{2x}$, and $y_3 = sin\, x$ independent? Show in 3 ways.

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Note: $y_1 = e^{rx}$, $y_2 = xe^{rx}$, $y_3 = x^2e^{rx}, \ldots$, $y_k = x^{k-1}e^{rx}$ are independent.

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Result: The set of all solutions to a homogeneous *n*-th order linear differential equation is a vector space of dimension *n*

This means…

1) You'll be able to find *n* independent solutions $y_1, y_2, \ldots, y_n$ to the DE
2) The general solution to the DE is $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$
3) Any independent set of *n* solutions to the DE will work

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Ex 6: $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = sin\,x$, and $y_4 = cos\,x$ are all independent solutions to the linear homogeneous differential equation $y^{(4)} - y = 0$. What is the general solution to the DE?

Definitions and Results Before We Begin Sections 4.2 and 4.3

Story

Result: The set of all solutions to a homogeneous *n*-th order linear differential equation is a vector space of dimension *n*

So the only question that remains is…

Question: How do we find n independent solutions to an n-th order linear differential equation?

Answer: On to sections 4.2 and 4.3 for the case where $n = 2$ and the DE has constant coefficients